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Recursion relations and branching rules for simple Lie algebras

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Abstract. The branching rules between simple Lie algebras and their regular (maximal) simple subalgebras are studied. Two types of recursion relations for anomalous relative multiplicities are obtained. One of them is proved to be the factorized version of the other. The factorization property is based on the existence of the set of weights Γ specific for each injection. The structure of Γ is easily deduced from the correspondence between the root systems of the algebra and subalgebra. The recursion relations thus obtained give rise to a simple and effective algorithm for branching rules. The details are illustrated by performing the explicit decomposition procedure for the injection $A_3 \oplus u(1) \rightarrow B_4$.

1. Introduction

1.1. In elementary particle physics and especially in model building it is quite important to have effective branching rules for Lie algebra representations. There are several simple methods of decomposition appropriate for different types of injections, for example, the Gelfand–Zeitlin method for $A_{n-1} \rightarrow A_n$, $B_{n-1} \rightarrow B_n$ and $D_{n-1} \rightarrow D_n$ [1, 2]. In the general case the most advanced investigation was performed by Moody and co-workers in a series of works [3–5]. Their approach is based on the properties of Weyl orbits in weight diagrams and the generating function technique [6].

In this paper we want to demonstrate that recursion relations for multiplicities of subrepresentations can also be successfully used in the decomposition procedure. We restrict the exposition to regular maximal injections of reductive subalgebras (composed of semisimple ones and diagonalizable Abelian algebras), although it is possible to treat analogously special injections and also non-maximal ones. For a simple algebra g and its regular maximal reductive subalgebra $\widetilde{g}$ the problem is to evaluate the coefficients n_μ in the decomposition of an irreducible representation $L^\lambda(g)$ (λ is its highest weight)

$$L^\lambda(g)_{\downarrow\widetilde{g}} = \oplus_\mu n_\mu \widetilde{L}^\mu(\widetilde{g}). \tag{1}$$

For any weight ν of L^λ the total multiplicity m_ν can be presented as the sum

$$m_\nu = m'_\nu + n_\nu \tag{2}$$

where m'_ν is the multiplicity induced by the subrepresentation $\oplus_{\mu>\nu} n_\mu \widetilde{L}^\mu(\widetilde{g})$ contained in (1). The second term n_ν in (2) is called *the relative multiplicity* of the weight ν. When ν is from the dominant Weyl chamber its relative multiplicity coincides with the corresponding

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coefficient in the decomposition (1). The recursion relations for relative multiplicities for the injection $A_3 \oplus u(1) \to D_4$ were studied in [7]. It was shown that the considerable set of multiplicities for intermediate weights mutually cancel and the final recursion formula is suitable for calculations. As will be demonstrated below the general recursion relation for regular injections can be formulated naturally in terms of *anomalous relative multiplicities* $\widetilde{n}_\nu$. To define them consider the highest weights

$$M = \{\mu \mid n_\mu \neq 0\}$$

for the decomposition (1). For the subalgebra $\widetilde{g}$ let V be the Weyl group and $\widetilde{\rho}$, the half-sum of the positive roots. The anomalous relative multiplicity $\widetilde{n}_\nu(g, \widetilde{g}, \lambda, \nu)$ is the function

$$\widetilde{n}(g, \widetilde{g}, \lambda, \nu) \equiv \widetilde{n}_\nu = \begin{cases} \det(v) n_\mu & \text{for } \{\mu \in M \mid v(\mu + \widetilde{\rho}) - \widetilde{\rho} = \nu\} \\ 0 & \text{elsewhere} \end{cases} \tag{3}$$

defined on the weight space of g. As will be demonstrated, the general recursion relation for regular injections can be naturally formulated in terms of anomalous relative multiplicities $\widetilde{n}_\nu$.

The paper is organized as follows. The general formalism is presented in section 2. The recursion relations thus obtained are based on the properties of the elementary 'fan' Γ, the special set of weights defined by the injection $\widetilde{g} \to g$. The structure of the Γ's and their basic properties are studied in detail. To demonstrate explicitly the role of Γ in the recursion procedure we use the very simple example of the injection $A_1 \oplus u(1) \to B_2$. Full details of the application of the recursion formulae to the injection $A_3 \oplus u(1) \to B_4$ are given in the appendix.

1.2. The notation used throughout the paper is as follows:

g: the simple Lie algebra;
$\widetilde{g}$: the reductive regular subalgebra of g;
Δ, $\widetilde{\Delta}$: the corresponding sets of positive roots (note that $\widetilde{\Delta}$ is the system of positive roots of the semisimple subalgebra in $\widetilde{g}$);
S, $\widetilde{S}$: the sets of basic roots;
ρ, $\widetilde{\rho}$: the half-sums of positive roots for g and $\widetilde{g}$, respectively;
W, V: the Weyl groups for Δ and $\widetilde{\Delta}$;
$\epsilon(w)$, $\epsilon(v)$: the determinants of the Weyl reflections w and v;
C, $\widetilde{C}$: the Weyl chambers dominant with respect to S and $\widetilde{S}$;
$\overline{C}$, $\overline{\widetilde{C}}$: the closures of the corresponding Weyl chambers;
P_g, $P_{\widetilde{g}}$: the weight lattices for g and $\widetilde{g}$;
$\mathcal{E}$, $\widetilde{\mathcal{E}}$: the formal unital associative algebras assigned to the weight lattices, $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are generated by the elements e^β, where β is the fundamental weight, and the composition $e^\beta \cdot e^\gamma = e^{\beta+\gamma}$;
$\mathrm{ch}\, L^\lambda$: the formal character of the representation L^λ;
Ψ^λ, $\widetilde{\Psi}^\mu$: the elements of formal algebras associated with the sets of anomalous weights for representations L^λ, $\widetilde{L}^\mu$, given by

$$\Psi^\lambda = \sum_{w \in W} \epsilon(w) \mathrm{e}^{w(\lambda+\rho)-\rho} \tag{4}$$

$$\widetilde{\Psi}^\mu = \sum_{v \in V} \epsilon(v) \mathrm{e}^{v(\mu+\widetilde{\rho})-\widetilde{\rho}}. \tag{5}$$

For roots and weights of simple Lie algebras we use the standard e-basis [9].

2. Recursion relations for regular injections

2.1. The initial decomposition (1) can be rewritten in terms of formal characters [11]

$$\mathrm{ch}\, L^{\lambda} = \sum_{\mu} n_{\mu}\, \mathrm{ch}\, L^{\mu}. \tag{6}$$

Applying the Weyl formula [10]

$$\mathrm{ch}\, L^{\xi} = \frac{\Psi^{\xi}}{\prod_{\alpha\in\Delta}(1-\mathrm{e}^{-\alpha})} \tag{7}$$

and taking into account the injection $\widetilde{\Delta} \to \Delta$ one obtains the relation between the anomalous elements Ψ^{λ} and $\widetilde{\Psi}^{\mu}$ as

$$\left(\prod_{\Delta\setminus\widetilde{\Delta}}(1-\mathrm{e}^{-\alpha})\right)^{-1}\Psi^{\lambda} = \sum_{\mu} n_{\mu}\widetilde{\Psi}^{\mu}. \tag{8}$$

Using the basis $\{e^{\xi}\}_{\xi\in P_g}$ of the algebra $\mathcal{E}$ one can expand both sides of relation (8). On the left-hand side of the expansion of the first factor

$$\left(\prod_{\Delta\setminus\widetilde{\Delta}}(1-\mathrm{e}^{-\alpha})\right)^{-1} = \sum_{\xi} K_{\widetilde{g}\subset g}(\xi)\mathrm{e}^{-\xi} \tag{9}$$

gives rise to the so-called Kostant–Heckman partition function [8]. Expression (9), together with the equations (4) and (5), gives the desired expansion of (8). Now consider only the weights μ from the dominant Weyl chamber $\widetilde{C}$, that is the anomalous weights with $v = e$. Comparing coefficients, one obtains the expression for the relative multiplicity n_{μ} in terms of the partition function $K_{\widetilde{g}\subset g}$:

$$n_{\mu} = \sum_{W} \epsilon(w) K_{\widetilde{g}\subset g}(w(\lambda+\rho)-(\rho+\mu)). \tag{10}$$

Note that relation (8) is valid on the whole weight lattice P_g. Thus one can rewrite it in the form:

$$\sum_{\mu} n_{\mu}\widetilde{\Psi}^{\mu} = \sum_{\mu} n_{\mu}\sum_{v\in V}\epsilon(v)\mathrm{e}^{v(\mu+\widetilde{\rho})-\widetilde{\rho}} = \sum_{\xi}\widetilde{n}_{\xi}\mathrm{e}^{\xi}. \tag{11}$$

Since all the weights $\{v(\mu+\widetilde{\rho})-\widetilde{\rho}\}$ are different the coefficients $\widetilde{n}_{\xi}$ here are just the anomalous relative multiplicities (see equation (3)). Relation (11), together with (9) and (4), shows that expression (10) is true in all points of P_g when n is changed to $\widetilde{n}$:

$$\widetilde{n}_{\xi} = \sum_{W} \epsilon(w) K_{\widetilde{g}\subset g}(w(\lambda+\rho)-(\rho+\xi)). \tag{12}$$

We will use this formula to construct the recursion relation for $\widetilde{n}_{\xi}$. First consider expression (12) for $\lambda = 0$. In this case $\widetilde{n}_{\xi}$ can be written explicitly as the multiplicity of the anomalous weight diagram for the trivial subrepresentation $\widetilde{L}^{0}$:

$$\widetilde{n}_{\xi} = \sum_{V}\epsilon(v)\delta_{v,\, v\widetilde{\rho}-\widetilde{\rho}} = \sum_{W}\epsilon(w)K_{\widetilde{g}\subset g}(w\rho-(\rho+\xi)). \tag{13}$$

One can extract the trivial term (with $w = e$) and obtain the recursion relation for the partition function $K_{\widetilde{g}\subset g}$

$$K_{\widetilde{g}\subset g}(\xi) = -\sum_{W\setminus e} K_{\widetilde{g}\subset g}(\xi+(w-1)\rho) + \sum_{V}\epsilon(v)\delta_{\xi,\rho-v\rho}. \tag{14}$$

Returning to expression (12), one easily comes to the conclusion that relation (14) induces the recursion relation for anomalous relative multiplicities

$$\widetilde{n}_\xi = -\sum_{W\setminus e} \epsilon(w)\widetilde{n}_{\xi+(1-w)\rho} + \sum_{W,V} \epsilon(w)\epsilon(v)\delta_{\xi+(1-v)\widetilde{\rho},\, w(\lambda+\rho)-\rho}\,. \tag{15}$$

This relation can be used in the explicit calculations of multiplicities $\widetilde{n}_\nu$ and n_μ and in some cases it is effective. But the necessity to estimate at each step the full set of probe weights $\{\xi + (1-w)\rho\}$ can make the whole process quite cumbersome. Consider once more equation (8). The previous derivation was based on the properties of the operator $\left(\prod_{\Delta\setminus\widetilde{\Delta}}(1-\mathrm{e}^{-\alpha})\right)^{-1}$. Now we shall use its inverse and taking into account (11) rewrite the relation (8) in the following form:

$$\Psi^\lambda = \sum_{w\in W} \epsilon(w)\mathrm{e}^{w(\lambda+\rho)-\rho} = \prod_{\Delta\setminus\widetilde{\Delta}}(1-\mathrm{e}^{-\alpha}) \cdot \sum_\xi \widetilde{n}_\xi \mathrm{e}^\xi. \tag{16}$$

The first factor in (16) defines the finite set of weights $\Gamma(\widetilde{g} \subset g)$ whose structure depends only on the injection $\widetilde{g} \subset g$,

$$\prod_{\Delta\setminus\widetilde{\Delta}}(1-\mathrm{e}^{-\alpha}) = 1 - \sum_{\gamma\in\Gamma} \mathrm{sign}(\gamma)\mathrm{e}^{-\gamma}. \tag{17}$$

In these terms equation (16) leads to the following recursion relation:

$$\widetilde{n}_\nu = \sum_{\gamma\in\Gamma} \mathrm{sign}(\gamma)\widetilde{n}_{\nu+\gamma} + \sum_{w\in W} \epsilon(w)\delta_{\nu,\, w(\lambda+\rho)-\rho}. \tag{18}$$

Its efficiency depends mainly on the possibility of constructing the set $\Gamma(\widetilde{g} \subset g)$ explicitly.

2.2. To reveal the structure of Γ let us use the denominator identity [10]

$$\prod_{\alpha\in\Delta}(1-\mathrm{e}^{-\alpha}) = \Psi^0 \tag{19}$$

to transform expression (17):

$$\prod_{\Delta\setminus\widetilde{\Delta}}(1-\mathrm{e}^{-\alpha}) = 1 - \sum_{\gamma\in\Gamma} \mathrm{sign}(\gamma)\mathrm{e}^{-\gamma} = \left(\prod_{\alpha\in\widetilde{\Delta}}(1-\mathrm{e}^{-\alpha})\right)^{-1} \cdot \Psi^0 \tag{20}$$

The anomalous element Ψ^0 is W-invariant and can be factorized with respect to $V \subset W$:

$$\Psi^0 = \sum_{x\in X}\sum_{v\in V} \epsilon(v\cdot x)\mathrm{e}^{(v\cdot x-1)\rho}. \tag{21}$$

Here X is the factor space W/V. This allows us to present Γ as the set of weight diagrams of representations $\widetilde{L}$:

$$\begin{aligned} 1 - \sum_{\gamma\in\Gamma} \mathrm{sign}(\gamma)\mathrm{e}^{-\gamma} &= \sum_{x\in X}\epsilon(x)\left(\prod_{\alpha\in\widetilde{\Delta}}(1-\mathrm{e}^{-\alpha})\right)^{-1} \sum_{v\in V}\epsilon(v)\mathrm{e}^{(vx\rho-\widetilde{\rho}+(\widetilde{\rho}-\rho))} \\ &= \mathrm{e}^{\widetilde{\rho}-\rho}\sum_{x\in X}\epsilon(x)\,\mathrm{ch}\,\widetilde{L}^{(x\rho-\widetilde{\rho})}\,. \end{aligned} \tag{22}$$

The element Ψ^0 multiplied by $(\prod_{\alpha\in\Delta}(1-\mathrm{e}^{-\alpha}))^{-1}$ gives the weight of the trivial representation L^0 of g, while the same Ψ^0 multiplied by $(\prod_{\alpha\in\widetilde{\Delta}}(1-\mathrm{e}^{-\alpha}))^{-1}$ generates the assembly $\Xi(\widetilde{g} \subset g)$ of weight diagrams for representations of $\widetilde{g}$. To construct $\Xi(\widetilde{g} \subset g)$ one can use the auxiliary set $\Omega(\widetilde{g} \subset g)$

$$\Omega = \{0, \alpha_{i_1}, \alpha_{i_1}+\alpha_{i_2}, \ldots, \alpha_{i_1}+\ldots+\alpha_{i_m} \mid \alpha_{i_l} \in \Delta\setminus\widetilde{\Delta}, m = \mathrm{card}(\Delta\setminus\widetilde{\Delta})\}.$$

Fix the subset Ω' of dominant weights

$$\Omega' = \{\omega \in \Omega \mid \omega \in \widetilde{\overline{C}}\}.$$

Equip every $\omega \in \Omega$ with the sign

$$\delta(\omega) = \delta(\alpha_{i_1} + \ldots + \alpha_{i_k}) = (-1)^{k+1}. \tag{23}$$

Reduce Ω' to Ω'_r cancelling every pair of weights in Ω' that has opposite signs. The subset of maximal weights in Ω'_r is just the desired assembly of representations $\Xi(\widetilde{g} \subset g)$ expressed in terms of their highest weights. Obviously each $\xi \in \Xi(\widetilde{g} \subset g)$ has the form $\xi = x\rho - \rho$ and it can easily be seen that $\delta(\xi) = \epsilon(x)$.

Let Φ^{ξ} be the weight diagram of the irrep $\widetilde{L}^{\xi}$ with the highest weight $\xi \in \Xi(\widetilde{g} \subset g)$. Due to the relation (22) the set $\Gamma(\widetilde{g} \subset g)$ can be obtained as the union of the diagrams Φ^{ξ}

$$\Gamma = \left(\bigcup_{\xi \in \Xi(\widetilde{g} \subset g)} \Phi^{\xi} \right) \Bigg\backslash \{0\}. \tag{24}$$

Thus to find $\Gamma(\widetilde{g} \subset g)$ it is sufficient to construct $\Xi(\widetilde{g} \subset g)$. As we have seen, the latter depends only on the structure of the factor space W/V and the weights ρ and $\widetilde{\rho}$.

Now we shall illustrate the situation when different types of regular maximal injections are treated. For the injection $A_{n-1} \oplus u(1) \to A_n$ the dimension of W/V gives $\mathrm{card}(\Xi) = (n+1)!/n! = n+1$. In the cases $A_{n-1} \oplus u(1) \to B_n, C_n, D_n$ the number of weights in Ξ is proportional to powers of '2':

$$\mathrm{card}(\Xi) = \begin{cases} 2^n & \text{for } A_{n-1} \oplus u(1) \to B_n \\ 2^n & \text{for } A_{n-1} \oplus u(1) \to C_n \\ 2^{n-1} & \text{for } A_{n-1} \oplus u(1) \to D_n. \end{cases}$$

The elementary analysis of the orbits of the Weyl group V on the space of faithful representation of W generated by the weight ρ leads to the following results.

Lemma 1. Put $\alpha_0 \equiv (0, \ldots, 0)$ and let $\{\alpha_1, \ldots, \alpha_s\}$ be the ordered sequences of roots:

$\{e_1 - e_{n+1}, e_2 - e_{n+1}, \ldots, e_n - e_{n+1}\}$ for $\Delta_{A_n} \setminus \Delta_{A_{n-1}}$

$\{e_1 + e_2, e_1 + e_3, \ldots, e_1 + e_n, e_1, e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_n\}$ for $\Delta_{B_n} \setminus \Delta_{B_{n-1}}$

$\{e_1 + e_2, e_1 + e_3, \ldots, e_1 + e_n, 2e_1, e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_n\}$ for $\Delta_{C_n} \setminus \Delta_{C_{n-1}}$

$\{e_1 + e_2, e_1 + e_3, \ldots, e_1 + e_n, e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_n\}$ for $\Delta_{D_n} \setminus \Delta_{D_{n-1}}$

then the set Ξ for $A_{n-1} \oplus u(1) \to A_n$ contains the weights

$$\xi_k = \sum_{j=0}^{k} \alpha_j \qquad k = 0, \ldots, n$$

while for $B_{n-1} \oplus u(1) \to B_n$ and $C_{n-1} \oplus u(1) \to C_n$

$$\xi_k = \sum_{j=0}^{k} \alpha_j \qquad k = 0, \ldots, 2n-1$$

and for $D_{n-1} \oplus u(1) \to D_n$

$$\xi_k = \sum_{j=0}^{k} \alpha_j \qquad k = 0, \ldots, 2n-2$$

$$\xi_{2n-1} = (n-1, 1, \ldots, 1, -1).$$

Lemma 2. In the sets $\Xi(A_{n-1} \oplus u(1) \to B_n, C_n, D_n)$ the zero weight is trivial

$$\xi_0 = (0, \ldots, 0)$$

the first weight has the form

$$\xi_1 = (p_1, p_2, \ldots, p_n) = \begin{cases} (1, 0, 0, \ldots, 0) & \text{for } A_{n-1} \oplus u(1) \to B_n \\ (2, 0, 0, \ldots, 0) & \text{for } A_{n-1} \oplus u(1) \to C_n \\ (1, 1, 0, \ldots, 0) & \text{for } A_{n-1} \oplus u(1) \to D_n \end{cases}$$

while the others can be obtained from the following relations:

$$\xi_{2^k} = (\underbrace{p_1 + k, 1, 1, \ldots, 1}_{k+l}, 0, \ldots, 0) \qquad l = \begin{cases} 1 & \begin{cases} \text{for } A_{n-1} \oplus u(1) \to B_n \\ \text{for } A_{n-1} \oplus u(1) \to C_n \end{cases} \\ 2 & \quad \text{for } A_{n-1} \oplus u(1) \to D_n \end{cases}$$

$$\xi_m = \xi_{2^k+i} = \xi_{2^k} + (\xi_i)_{\text{shift}} \qquad i = 1, \ldots, 2^k - 1$$

where for every $\xi_i = (q_1, q_2, \ldots, q_n)$ the shifted weight $(\xi_i)_{\text{shift}}$ is defined with the coordinates

$$(\xi_i)_{\text{shift}} = (0, q_1, q_2, \ldots, q_{n-1}).$$

Note that in the framework of rules described in lemmas 1 and 2 the sets Ξ are totally defined by the first weight ξ_1.

To conclude the general exposition of the recursion properties of $\widetilde{n}_\mu$ we show the interdependence of the two recursion formulae (15) and (18).

Lemma 3. The recursion relation (15) can be factorized with respect to the subgroup V of W so that the summation over the factor space $W \setminus V$ is replaced by the summation over Γ.

Proof. Use equation (18) to write down the recursion relation for the expression $\sum_{v \in V} \epsilon(v) \widetilde{n}_{\nu+(1-v)\widetilde{\rho}}$ as a whole and extract the first term corresponding to $v = e$:

$$\begin{aligned} \widetilde{n}_\nu = &- \sum_{v \in V, v \neq e} \epsilon(v) \widetilde{n}_{\nu+(1-v)\widetilde{\rho}} - \sum_{v \in V} \sum_{\gamma \in \Gamma} \epsilon(v) \operatorname{sign}(\gamma) \widetilde{n}_{\nu+(1-v)\widetilde{\rho}+\gamma} \\ &+ \sum_{w \in W, v \in V} \epsilon(v)\epsilon(w) \delta_{\nu+(1-v)\widetilde{\rho},\, w(\lambda+\rho)-\rho} \\ = &\sum_{V, \Gamma \cup \{0\}; (v,\gamma) \neq (e,0)} \epsilon(v) \operatorname{sign}(\gamma) \widetilde{n}_{\nu+(1-v)\widetilde{\rho}+\gamma} \end{aligned}$$

$$+ \sum_{w \in W, v \in V} \epsilon(v)\epsilon(w)\delta_{v+(1-v)\widetilde{\rho},\ w(\lambda+\rho)-\rho}. \tag{25}$$

Comparing this expression with (15) we see that by introducing Γ one provides the factorization in the first term of relation (15). Thus relation (18) can be called *the factorized recursion formula* for anomalous relative multiplicities. □

In table 1 we bring together the information about the Ξ's for the types of injections described in this paper.

2.3. Both equations (15) and (18) provide effective tools to treat the branching rules decompositions for maximal regular injections. One can easily estimate the relative capacities of these relations for different pairs of g and $\widetilde{g}$. The result is that there are five families of injections mostly favourable for relation (1): $A_{n-1} \oplus u(1) \to A_n$, $B_{n-1} \oplus u(1) \to B_n$, $C_{n-1} \oplus u(1) \to C_n$, $D_{n-1} \oplus u(1) \to D_n$, $A_{n-1} \oplus u(1) \to B_n$. For these five types the efficiency of equation (18) increases with the growth in n compared with that of (15). For the first four types the ordinary decomposition methods are suitable (the Gelfand–Zeitlin procedure [1], for example). So we shall concentrate our attention on the last family: $A_{n-1} \oplus u(1) \to B_n$.

To show the application of the factorized formula (18) in detail we shall start with a quite simple example. Consider the injection $A_1 \oplus u(1) \to B_2$. Fix the basic roots of B_2:

$$S(B_2) = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2\}$$

and the fundamental weights:

$$\{\beta_1 = e_1, \beta_2 = \tfrac{1}{2}(e_1 + e_2)\}.$$

According to table 1 we have four highest weights in the set $\Xi(A_1 \oplus u(1) \to B_2)$:

$$\Xi(A_1 \oplus u(1) \to B_2) = \{(0,0), (1,0), (2,1), (2,2)\}$$

Thus the set $\Gamma(A_1 \oplus u(1) \to B_2)$ contains the weight diagrams of the $A_1 \oplus u(1)$ representations $\Phi^{\xi_1} = ([1], 1)$, $\Phi^{\xi_2} = ([1], 3)$, $\Phi^{\xi_s} = ([0], 4)$. (The $u(1)$ generator is normalized to have integer eigenvalues.)

$$\Gamma = \left(\bigcup_{\xi \in \Xi} \Phi^{\xi} \right) \Bigg\backslash \{0\} = \{\gamma^{(\mathrm{sign}\,\gamma)}\} = \{(1,0)^{(+)}, (0,1)^{(+)}, (2,1)^{(-)}, (1,2)^{(-)}, (2,2)^{(+)}\}.$$

To simplify the following steps it is convenient to perform further splitting of the diagram of anomalous weights for the subrepresentations $\widetilde{L}^{\mu}$. This splitting is not unique; one can choose an arbitrary vector $\varepsilon \in C$ and the projections $a(\kappa)$ are obtained as the scalar products

$$\langle (\kappa - \lambda), -\varepsilon \rangle = a(\kappa)$$

for every κ from to the weight lattice P_g. The weight κ is said to belong to the level $a(\kappa)$. The ordering for the components in Ψ^{λ} thus induced guarantees an unambiguous level by level application of the recursion formula (18). If $\Delta \setminus \widetilde{\Delta}$ contains no positive roots orthogonal to the boundary of $\overline{C}$, the auxiliary vector ε may be placed in the closure of C as well.

Consider the irreducible representation L^{λ} of B_2 with the highest weight $\lambda = (\frac{5}{2}, \frac{1}{2})$. The corresponding anomalous weight diagram Ψ^{λ} contains eight vectors:

$$\Psi^{(\frac{5}{2},\frac{1}{2})} = \{\psi^{\epsilon(w)}\} = \left\{ (\tfrac{5}{2}, \tfrac{1}{2})^{(+)}, (-\tfrac{1}{2}, \tfrac{7}{2})^{(-)}, (\tfrac{5}{2}, -\tfrac{3}{2})^{(-)}, (-\tfrac{5}{2}, \tfrac{7}{2})^{(+)}, (-\tfrac{1}{2}, -\tfrac{9}{2})^{(+)}, \right.$$

$$\left. (-\tfrac{11}{2}, \tfrac{1}{2})^{(-)}, (-\tfrac{5}{2}, -\tfrac{9}{2})^{(-)}, (-\tfrac{11}{2}, -\tfrac{3}{2})^{(+)} \right\}.$$

Table 1.

	The set $\Xi(\widetilde{g} \subset g)$ in terms of:		
$\widetilde{g} \to g$	highest weights ξ	Dynkin indices of $\widetilde{L}^{\xi}$	sign $\gamma(\xi)$
$A_{n-1} \oplus u(1) \to A_n$	$(0, 0, \ldots, 0)$	$([0, 0, \ldots, 0], 0)$	
	$(1, 0, \ldots, 0, -1)$	$([1, 0, \ldots, 0], 1)$	$+$
	$(1, 1, 0, \ldots, 0, -2)$	$([0, 1, \ldots, 0], 2)$	$-$
	$\vdots$	$\vdots$	$\vdots$
	$(1, 1, \ldots, 1, -n)$	$([0, 0, \ldots, 0], n)$	$(-1)^{n+1}$
$A_{n-1} \oplus u(1) \to B_n$	$(0, 0, \ldots, 0)$	$([0, 0, \ldots, 0], 0)$	
	$(1, 0, \ldots, 0)$	$([1, 0, \ldots, 0], 1)$	$+$
	$(2, 1, 0, \ldots, 0)$	$([1, 1, 0, \ldots, 0], 3)$	$-$
	$(2, 2, 0, \ldots, 0)$	$([0, 2, 0, \ldots, 0], 4)$	$+$
	$(3, 1, 1, 0, \ldots, 0)$	$([2, 0, 1, 0, \ldots, 0], 5)$	$+$
	$(3, 2, 1, 0, \ldots, 0)$	$([1, 1, 1, 0, \ldots, 0], 6)$	$-$
	$(3, 3, 2, 0, \ldots, 0)$	$([0, 1, 2, 0, \ldots, 0], 8)$	$+$
	$(3, 3, 3, 0, \ldots, 0)$	$([0, 0, 3, 0, \ldots, 0], 9)$	$-$
	$\vdots$	$\vdots$	$\vdots$
	$(n, n, \ldots, n, n-1)$	$([0, 0, \ldots, 0, 1], n^2 - 1)$	$(-1)^{1/2(n^2+n)}$
	$(n, n, \ldots, n)$	$([0, 0, \ldots, 0], n^2)$	$(-1)^{1/2(n^2+n+2)}$
$A_{n-1} \oplus u(1) \to C_n$	$(0, 0, \ldots, 0)$	$([0, 0, \ldots, 0], 0)$	
	$(2, 0, \ldots, 0)$	$([2, 0, \ldots, 0], 2)$	$+$
	$(3, 1, 0, \ldots, 0)$	$([2, 1, 0, \ldots, 0], 4)$	$-$
	$(3, 3, 0, \ldots, 0)$	$([0, 3, 0, \ldots, 0], 6)$	$+$
	$(4, 1, 1, 0, \ldots, 0)$	$([3, 0, 1, 0, \ldots, 0], 6)$	$+$
	$(4, 3, 1, 0, \ldots, 0)$	$([1, 2, 1, 0, \ldots, 0], 8)$	$-$
	$(4, 4, 2, 0, \ldots, 0)$	$([0, 2, 2, 0, \ldots, 0], 10)$	$+$
	$(4, 4, 4, 0, \ldots, 0)$	$([0, 0, 4, 0, \ldots, 0], 12)$	$-$
	$\vdots$	$\vdots$	$\vdots$
	$(n+1, \ldots, n+1, n-1)$	$([0, 0, \ldots, 2], n^2 + n - 2)$	$(-1)^{1/2(n^2+n)}$
	$(n+1, \ldots, n+1)$	$([0, 0, \ldots, 0], n^2 + n)$	$(-1)^{1/2(n^2+n+2)}$
$A_{n-1} \oplus u(1) \to D_n$	$(0, 0, \ldots, 0)$	$([0, 0, \ldots, 0], 0)$	
	$(1, 1, 0, \ldots, 0)$	$([0, 1, 0, \ldots, 0], 2)$	$+$
	$(2, 1, 1, 0, \ldots, 0)$	$([1, 0, 1, 0, \ldots, 0], 4)$	$-$
	$(2, 2, 2, 0, \ldots, 0)$	$([0, 0, 2, 0, \ldots, 0], 6)$	$+$
	$(3, 1, 1, 1, 0, \ldots, 0)$	$([2, 0, 0, 1, 0, \ldots, 0], 6)$	$+$
	$(3, 2, 2, 1, 0, \ldots, 0)$	$([1, 0, 1, 1, 0, \ldots, 0], 8)$	$-$
	$(3, 3, 2, 2, 0, \ldots, 0)$	$([0, 1, 0, 2, 0, \ldots, 0], 10)$	$+$
	$(3, 3, 3, 3, 0, \ldots, 0)$	$([0, 0, 0, 3, 0, \ldots, 0], 12)$	$-$
	$\vdots$	$\vdots$	$\vdots$
	$(n-1, \ldots, n-1, n-2, n-2)$	$([0, \ldots, 0, 1, 0], n^2-n-2)$	$(-1)^{1/2(n^2-n)}$
	$(n-1, \ldots, n-1)$	$([0, \ldots, 0], n^2 - n)$	$(-1)^{1/2(n^2-n+2)}$

This is the case when splitting can be simplified. One can choose $\varepsilon = (1, 1) \in \overline{C}$ so that $\langle \kappa, \varepsilon \rangle$ becomes proportional to the eigenvalues of the $u(1)$ generator in the L^{λ} representation. Applying the factorized formula to obtain the decomposition of L^{λ} we are interested in κ's within the Weyl chamber $\overline{\widetilde{C}}$. Thus for our example only the weights with non-negative projection on $\alpha_1 = (1, -1)$ may have positive multiplicities n_{κ}. At the zeroth level the

Table 1. (Continued)

	The set $\Xi(\widetilde{g} \subset g)$ in terms of:		
$\widetilde{g} \to g$	highest weights ξ	Dynkin indices of $\widetilde{L}^{\xi}$	sign $\gamma(\xi)$
$B_{n-1} \oplus u(1) \to B_n$	$(n-2, 1, 1, \ldots, 1, 0)$	$(0, 0, \ldots, 0)$	$([0, 0, \ldots, 0], 0)$
	$(1, 1, 0, \ldots, 0)$	$([1, 0, \ldots, 0], 1)$	$+$
	$(2, 1, 1, 0, \ldots, 0)$	$([0, 1, 0, \ldots, 0], 2)$	$-$
	$(3, 1, 1, 1, 0, \ldots, 0)$	$([0, 0, 1, 0, \ldots, 0], 3)$	$+$
	$\vdots$	$\vdots$	$\vdots$
	$([0, 0, \ldots, 1, 0], n-2)$	$(-1)^{n-1}$	
	$(n-1, 1, 1, \ldots, 1)$	$([0, \ldots, 0, 2], n-1)$	$(-1)^{n}$
	$(n, 1, 1, \ldots, 1)$	$([0, \ldots, 0, 2], n)$	$(-1)^{n+1}$
	$(n+1, 1, 1, \ldots, 1, 0)$	$([0, \ldots, 1, 0], n+1)$	$(-1)^{n+2}$
	$\vdots$	$\vdots$	$\vdots$
	$(2n-2, 1, 0, \ldots, 0)$	$([1, 0, \ldots, 0], 2n-2)$	$-$
	$(2n-1, 0, \ldots, 0)$	$([0, 0, \ldots, 0], 2n-1)$	$+$
$C_{n-1} \oplus u(1) \to C_n$	$(0, 0, \ldots, 0)$	$([0, 0, \ldots, 0], 0)$	
	$(1, 1, 0, \ldots, 0)$	$([1, 0, \ldots, 0], 1)$	$+$
	$(2, 1, 1, 0, \ldots, 0)$	$([0, 1, 0, \ldots, 0], 2)$	$-$
	$(3, 1, 1, 1, 0, \ldots, 0)$	$([0, 0, 1, 0, \ldots, 0], 3)$	$+$
	$\vdots$	$\vdots$	$\vdots$
	$(n-1, 1, \ldots, 1)$	$([0, \ldots, 0, 1], n-1)$	$(-1)^{n}$
	$(n+1, 1, \ldots, 1)$	$([0, \ldots, 0, 1], n+1)$	$(-1)^{n+1}$
	$(n+2, 1, \ldots, 1, 0)$	$([0, \ldots, 0, 1, 0], n+2)$	$(-1)^{n+2}$
	$\vdots$	$\vdots$	$\vdots$
	$(2n-1, 1, 0, \ldots, 0)$	$([1, 0, \ldots, 0], 2n-1)$	$-$
	$(2n, 0, \ldots, 0)$	$([0, \ldots, 0], 2n)$	$+$
$D_{n-1} \oplus u(1) \to D_n$	$(n-3, 1, \ldots, 1, 0, 0)$	$(0, 0, \ldots, 0)$	$([0, 0, \ldots, 0], 0)$
	$(1, 1, 0, \ldots, 0)$	$([1, 0, \ldots, 0], 1)$	$+$
	$(2, 1, 1, 0, \ldots, 0)$	$([0, 1, 0, \ldots, 0], 2)$	$-$
	$\vdots$	$\vdots$	$\vdots$
	$([0, \ldots, 0, 1, 0, 0], n-3)$	$(-1)^{n-2}$	
	$(n-2, 1, \ldots, 1, 0)$	$([0, \ldots, 0, 1, 1], n-2)$	$(-1)^{n-1}$
	$(n-1, 1, \ldots, 1, -1)$	$([0, \ldots, 0, 2, 0], n-1)$	$(-1)^{n}$
	$(n-1, 1, \ldots, 1, 1)$	$([0, \ldots, 0, 2], n-1)$	$(-1)^{n}$
	$(n, 1, \ldots, 1, 0)$	$([0, \ldots, 0, 1, 1], n)$	$(-1)^{n+1}$
	$(n+1, 1, \ldots, 1, 0, 0)$	$([0, \ldots, 1, 0, 0], n+1)$	$(-1)^{n+2}$
	$\vdots$	$\vdots$	$\vdots$
	$(2n-3, 1, 0, \ldots, 0)$	$([1, 0, \ldots, 0], 2n-3)$	$+$
	$(2n-2, 0, \ldots, 0)$	$([0, \ldots, 0], 2n-2)$	$-$

result is trivial:

$$\widetilde{n}_{(\frac{5}{2}, \frac{1}{2})} = n_{(\frac{5}{2}, \frac{1}{2})} = 1$$

$$\widetilde{n}_{(-\frac{1}{2}, \frac{7}{2})} = -1.$$

At the next level (called the first) one finds two points in $\widetilde{\widetilde{C}}$, $(\frac{5}{2}, -\frac{1}{2})$ and $(\frac{3}{2}, \frac{1}{2})$, where equation (18) gives non-zero values for $\widetilde{n}_\kappa$.

$$n_{(\frac{5}{2},-\frac{1}{2})} = 1 \qquad n_{(\frac{3}{2},\frac{1}{2})} = 1.$$

Similarly on the following two levels one finds

$$n_{(\frac{1}{2},\frac{1}{2})} = 1 \qquad n_{(\frac{3}{2},-\frac{1}{2})} = 2$$

and

$$n_{(\frac{1}{2},-\frac{1}{2})} = 2 \qquad n_{(\frac{3}{2},-\frac{3}{2})} = 1.$$

Due to the (reflection) symmetry of the weight diagram these four levels give sufficient information to write down the final result:

$$[2,1]_{\downarrow A_1\oplus u(1)} = ([2],3) \oplus ([1],2) \oplus ([3],2) \oplus 2([2],1) \oplus ([0],1) \oplus 2([1],0) \oplus ([3],0)$$
$$\oplus 2([2],-1) \oplus ([0],-1) \oplus ([3],-2) \oplus ([1],-2) \oplus ([2],3).$$

Here the numbers in the square brackets are the Dynkin indices and the last term in parenthesis is the eigenvalue of the $u(1)$ generator. Note that in performing this reduction we do not need to take into account the anomalous weights outside the dominant chamber $\widetilde{\widetilde{C}}$. Such additional decomposition of the recurrence property occurs only when the vectors of the form $\{\gamma + \xi \mid \xi \in \widetilde{\widetilde{C}}, \gamma \in \Gamma\}$ do not reach the domain of anomalous weights of $\widetilde{g}$ in $P_g \setminus (P_g \cap \widetilde{\widetilde{C}})$.

In [7] an attempt was made to achieve the additional decomposition of the recurrence property in the situation when the previous condition fails. The injection $A_3 \oplus u(1) \to D_4$ was studied and the recurrence relation connecting only relative multiplicities was obtained. It is slightly different from that described by equation (18). However, one faces great difficulties in trying to obtain such an algorithm for other pairs of algebras.

In the appendix we give a more complicated example by considering the injection $A_3 \oplus u(1) \to B_4$. This demonstrates the efficiency of the decomposition algorithm based on the factorized recurrence formula (18). The whole computation is relatively simple and can easily be computerized. The non-maximal regular injections and special injections can be treated similarly. The detailed study of recurrence relations in these cases will be presented in a forthcoming publication.

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Appendix. The injection $A_3 \oplus u(1) \to B_4$

The positive root systems for this maximal regular injection in the standard e-basis can be written as follows:

$$\left.\begin{aligned} &\Delta(B_4) = \{e_i, e_j - e_k, e_j + e_k\} \\ &\widetilde{\Delta}(A_3) = \{e_j - e_k\} \\ &\Delta \setminus \widetilde{\Delta} = \{e_i, e_j + e_k\} \end{aligned}\right\} \qquad i, j, k = 1, \ldots, 4 \quad j < k.$$

According to lemma 2 the set Ξ contains 16 irreducible representations of $A_3 \oplus u(1)$ enumerated by their highest weights ξ_i $(i = 0, \ldots, 15)$:

$$\begin{aligned}
&\Xi(A_3 \oplus u(1) \rightarrow B_4) \\
&\quad = \{(0,0,0,0); (1,0,0,0); (2,1,0,0); (2,2,0,0); (3,1,1,0); (3,2,1,0); \\
&\qquad (3,3,2,0); (3,3,3,0); (4,1,1,1); (4,2,1,1); (4,3,2,1); (4,3,3,1); \\
&\qquad (4,4,2,2); (4,4,3,2); (4,4,4,3); (4,4,4,4)\}\,.
\end{aligned}$$

In the 'fan' $\Gamma(A_3 \oplus u(1) \rightarrow B_4)$ the weights γ for each Φ^{ξ_s} bear the same sign, thus the signs can be attributed to the representations $\widetilde{L}^{\xi_s}$:

$$\begin{aligned}
&\{\widetilde{L}^{\xi_s}; \operatorname{sign}(\gamma)\}_{s=1,\ldots,15} \\
&\quad = \{(([1,0,0],1); (+)), (([1,1,0],3); (-)), (([0,2,0],4); (+)), \\
&\qquad (([2,0,1],5); (+)), (([1,1,1],6); (-)), (([3,0,0],7); (-)), \\
&\qquad (([0,1,2],8); (+)), (([2,1,0],8); (+)), (([0,0,3],9); (-)), \\
&\qquad (([1,1,1],10); (-)), (([1,0,2],11); (+)), (([0,2,0],12); (+)), \\
&\qquad (([0,1,1],13); (-)), (([0,0,1],15); (+)), (([0,0,0],16); (-))\}\,.
\end{aligned}$$

The non-trivial multiplicities of $\gamma \in \Gamma$ must also be taken into account. To obtain the splitting one can choose the vector $\varepsilon = (1,1,1,1)$. We write down explicitly only those weights γ that describe the first and the second levels of decomposition:

$$\begin{aligned}
&\Gamma(A_3 \oplus u(1) \rightarrow B_4) = \bigcup_{\xi_s, s=1,\ldots,15} \Phi^{\xi_s;\, \operatorname{sign}(\gamma)} \\
&\quad = \big\{(1,0,0,0)^{(+)}, (0,1,0,0)^{(+)}, (0,0,1,0)^{(+)}, (0,0,0,1)^{(+)}, (2,1,0,0)^{(-)}, \\
&\qquad (2,0,1,0)^{(-)}, (2,0,0,1)^{(-)}, (1,0,0,2)^{(-)}, (0,1,0,2)^{(-)}, (0,0,1,2)^{(-)}, \\
&\qquad (1,2,0,0)^{(-)}, (0,2,1,0)^{(-)}, (0,2,0,1)^{(-)}, (1,0,2,0)^{(-)}, (0,1,2,0)^{(-)}, \\
&\qquad (0,0,2,1)^{(-)}, 2(1,1,1,0)^{(-)}, 2(1,1,0,1)^{(-)}, 2(1,0,1,1)^{(-)}, 2(0,1,1,1)^{(-)}, \\
&\qquad \ldots, (4,4,4,4)^{(-)}\big\}\,.
\end{aligned}$$

Consider, for example, the irrep $L^{\lambda}(B_4)$ with $\lambda = (2,2,1,0)$; $\dim L^{\lambda} = (\underline{1650})$. It has 11 levels. Due to the reflection symmetry $((k_1,k_2,k_3,k_4) \leftrightarrow (-k_1,-k_2,-k_3,-k_4))$ of the weight diagram it is sufficient to study only six of them.

In contrast to the previous case, the recurrence procedure cannot be performed separately for the relative multiplicities n_κ, that is for $\widetilde{n}_\kappa$ with $\kappa \in \overline{\widetilde{C}}$. Nevertheless, the calculations involving the anomalous weights (in other Weyl chambers) can be simplified considerably thanks to the following two considerations:

- In the level-by-level recursive procedure, after the evaluation of $\widetilde{n}$ in $\overline{\widetilde{C}}$, the anomalous relative weights in other points can be obtained using the Weyl group V.
- Using the exterior contour of Γ one can easily fix the domain of the weight space that can contribute to the relative multiplicities n_κ and pay no attention to the weights outside this domain.

For the injection $A_3 \oplus u(1) \to B_4$ only the anomalous weights with non-negative first coordinate must be taken into account to obtain $\widetilde{n} \in \widetilde{\overline{C}}$. Thus for every highest weight μ obtained one must find only twelve anomalous points of Ψ^{μ} to be able to carry the recursion to the next level. At the zeroth level, besides the highest weight

$$n_{(2,2,1,0)} = \widetilde{n}_{(2,2,1,0)} = 1$$

one must also calculate (using the Weyl group V) the anomalous relative multiplicities:

$$\widetilde{n}_{(2,2,-1,2)} = -1 \qquad \widetilde{n}_{(2,0,3,0)} = -1 \qquad \widetilde{n}_{(2,0,-1,4)} = 1 \qquad \widetilde{n}_{(2,-2,3,2)} = 1$$

$$\widetilde{n}_{(2,-2,1,4)} = -1 \qquad \widetilde{n}_{(1,3,1,0)} = -1 \qquad \widetilde{n}_{(1,3,-1,2)} = 1 \qquad \widetilde{n}_{(1,0,4,0)} = 1$$

$$\widetilde{n}_{(1,0,-1,5)} = -1 \qquad \widetilde{n}_{(1,-2,4,2)} = -1 \qquad \widetilde{n}_{(1,-2,1,5)} = 1.$$

After this equation (18) can be applied directly to obtain the anomalous relative multiplicities at the first level, among them being

$$n_{(2,2,0,0)} = 1 \qquad n_{(2,1,1,0)} = 1.$$

The 24 anomalous points of these two representations fix the decomposition at the second level:

$$n_{(2,1,0,0)} = 2 \qquad n_{(1,1,1,0)} = 1 \qquad n_{(2,1,1,-1)} = 1 \qquad n_{(2,2,0,-1)} = 1$$

and so on. For example, at the third level equation (18) leads to the following relation for the weight $(1,1,0,0)$:

$$\widetilde{n}_{(1,1,0,0)} = n_{(1,1,0,0)} = n_{(1,1,1,0)} + n_{(2,1,0,0)} - 2n_{(2,2,1,0)} - \widetilde{n}_{(1,3,1,0)} = 2.$$

The final result is

$$\begin{aligned}
[0,1,1,0]_{\downarrow A_3 \oplus u(1)} = {} & ([0,1,1],5) \oplus ([0,2,0],4) \oplus ([1,0,1],4) \oplus 2([1,1,0],3) \\
& \oplus ([0,0,1],3) \oplus ([1,0,2],3) \oplus ([0,2,1],3) \oplus ([2,0,0],2) \\
& \oplus 2([0,1,0],2) \oplus ([0,0,2],2) \oplus 2([1,1,1],2) \oplus 2([1,0,0],1) \\
& \oplus 2([2,0,1],1) \oplus 3([0,1,1],1) \oplus ([1,1,2],1) \oplus ([1,2,0],1) \\
& \oplus ([0,0,0],0) \oplus 3([1,0,1],0) \oplus 2([0,2,0],0) \oplus ([2,0,2],0) \\
& \oplus ([2,1,0],0) \oplus ([0,1,2],0) \oplus 2([1,0,2],-1) \oplus 2([0,0,1],-1) \\
& \oplus 2([1,1,1],-2) \oplus ([2,0,0],-2) \oplus 2([0,1,0],-2) \oplus ([0,0,2],-2) \\
& \oplus ([1,2,0],-3) \oplus ([2,0,1],-3) \oplus ([1,0,0],-3) \oplus 2([0,1,1],-3) \\
& \oplus ([1,0,1],-4) \oplus ([0,2,0],-4) \oplus ([1,1,0],-5).
\end{aligned}$$

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